

Deformation Quantization in Singular Spaces

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Abstract

We present a method of quantizing analytic spaces X immersed in an arbitrary smooth ambient manifold M . Remarkably our approach can be applied to singular spaces. We begin by quantizing the cotangent bundle of the manifold M . Using a super-manifold framework we modify the Fedosov construction in a way such that the \star -product of the functions lifted from the base manifold turns out to be the usual commutative product of smooth functions on M . This condition allows us to lift the ideals associated to the analytic spaces on the base manifold to form left (or right) ideals on $(\mathcal{O}_{\Omega^1 M}[[\hbar]], \star_{\hbar})$ in a way independent of the choice of generators and leading to a finite set of PDEs defining the functions in the quantum algebra associated to X . Some examples are included.

1 Introduction

Deformation quantization is mathematically speaking a way of defining non-commutative associative products on a Poisson manifold, called \star -products, in a way such that the non-commutativity is controlled by a deformation parameter. The usual pointwise product is recovered as a limit case when this deformation parameter is negligible and the Poisson structure is found to be in the same fashion a limit case of the \star -commutator in accordance with the quantum correspondence principle.

Formally speaking, consider a smooth manifold N . A star-product on $\mathcal{O}_N[[\hbar]]$ is an associative $\mathbb{R}[[\hbar]]$ -linear product

$$f \star_{\hbar} g := \sum_{k=0}^{\infty} \left(-\frac{i\hbar}{2}\right)^k m_k(f, g)$$

where any m_k is a bidifferential operator of finite total order and $m_0(f, g) = fg$. It then can be shown that the operation on $\mathcal{O}_N[[\hbar]]$ defined by $\{f, g\} = \lim_{\hbar \rightarrow 0} \frac{1}{i\hbar} [f, g]$ is indeed a Poisson structure on M . (For a review from the mathematical perspective see [1].)

From the physics point of view deformation quantization is a new autonomous reformulation of quantum mechanics. Although still in development nowadays it is capable of reproducing numerous examples from the ordinary operator formulation and has been found to be closely related to the path integral formulation. (For a review from the physics perspective see [2] [3].)

One of the most powerful trends of work in mathematical physics during the last century was the generalization of the formulation of physical theories from the Euclidean case to the manifold framework. In this way classical mechanics was formulated in terms of a Poisson structure generalizing the classical notion of the Poisson bracket. Another example of this has been the generalization of the operator formulation of quantum mechanics to the non-flat case. Different answers have been found e.g. geometric quantization, group theoretic quantization. (For an extended discussion see [4].)

Deformation quantization has followed a similar way going from the first \star -product found by Moyal following physical ideas and then generalized by Fedosov, who gave an explicit construction of a \star -product in an arbitrary symplectic manifold [5] [6].

The quantization of singular spaces has been somewhat rejected until very recently. This is one of the new possibilities provided by deformation quantization, since the traditional approaches breakdown in this case. Merkulov proposed in [7] a way of quantizing algebraic varieties immersed in some \mathbb{R}^n . This included the possibility of a non-empty set of singular points.

In this work we provide a general construction to quantize arbitrary analytic spaces (including the singular case), immersed in any smooth analytic real manifold. (A different approach can be found in [8]). We summarize the general context of this work in the following table.

	Euclidean	Manifold framework
Classical Mechanics	Poisson bracket	Poisson structure
Quantum Mechanics.	Heisenberg's formulation	e.g. group quantization
Deformation Quantization	Moyal \star -product	Fedosov construction
Deformation Quantization of Singular spaces	Merkulov's work	This article

2 Quantization of Analytic Spaces

Consider a smooth analytic manifold M and $\mathcal{O}_M[[\hbar]]$, the ring of formal power series with global analytic functions as coefficients equipped with the usual commutative product. Classically an analytic space $(X, \mathcal{O}_X[[\hbar]])$ immersed on M is defined by choosing a finitely generated vanishing ideal I . Then the subspace X , which in general is not a smooth submanifold, corresponds to the set of solutions of the system of equations $\phi_i = 0$, for any set ϕ_1, \dots, ϕ_n of generators of I . The associated ring of functions is defined then as $\mathcal{O}_X[[\hbar]] = \mathcal{O}_M[[\hbar]]/I$. (For a detailed exposition see for example [9]).

Now consider the cotangent bundle $\pi : \Omega^1 M \rightarrow M$. The ideal I can be lifted via $\pi^{-1}(I)$ and in this way an analytic space $(X, \mathcal{O}_X[[\hbar]])$ can be defined. This time $\mathcal{O}_X[[\hbar]] = \mathcal{O}_{\Omega^1 M}[[\hbar]]/\pi^{-1}(I)$ and $X \subset \Omega^1 M$. In physical terms this would describe a system with a set of constrictions in the configuration space. Our goal is to define a quantum version of this structure capable of dealing with singular spaces.

Note that if one tries to replace naively the pointwise product of functions by the star-product on $\Omega^1 M$ (which can be found via the standard Fedosov construction for symplectic manifolds), one finds two possible scenarios. If one fixes a set of generators of the vanishing ideal $\pi^{-1}(I)$ before quantizing, the construction will depend on this choice since the star product depends on the smoothness around the vanishing points of the possible choices of generators. On the other hand if one avoids this choice and defines a left (or right) ideal and proceeds to determine the normalizer, one finds an infinite number of equations.

In our approach, we modify the Fedosov construction [5] [6] in a way that allow us to find a quantum algebra independently of the choice of generators and corresponding to the solutions of a finite set of partial differential equations.

The quantization procedure goes as follows

Step 1 *Initial data.* *The necessary data to begin the construction is:*

1. *A smooth manifold M .*
2. *A torsion free affine connection ∂ defined on M .*
3. *The vanishing ideal I associated to the classical analytic space.*

Step 2 *Quantization of $\Omega^1 M$.*

Before discussing the details¹ we give in rough terms an overview of this step. The goal is to construct a star product on the ring of functions $\mathcal{O}_{\Omega^1 M}[[\hbar]]$ which coincides with the pointwise multiplication when restricted to functions lifted from the base, which is not the case in general for the standard Fedosov construction. The main tool is an auxiliary algebra \mathcal{W} where a star product \star_{\hbar} can be defined in a straightforward manner. Then a subalgebra $\mathcal{W}_D \subset \mathcal{W}$ with a one to one correspondence that we will denote as $\phi : \mathcal{O}_{\Omega^1 M}[[\hbar]] \rightarrow \mathcal{W}_D$ is found. The star product \star'_{\hbar} for $\mathcal{O}_{\Omega^1 M}[[\hbar]]$ is defined as the one making the following diagram commute

$$\begin{array}{ccc} \mathcal{O}_{\Omega^1 M}[[\hbar]] \otimes \mathcal{O}_{\Omega^1 M}[[\hbar]] & \xrightarrow{\star'_{\hbar}} & \mathcal{O}_{\Omega^1 M}[[\hbar]] \\ \phi \otimes \phi \downarrow & & \downarrow \phi \\ \mathcal{W}_D \otimes \mathcal{W}_D & \xrightarrow{\star_{\hbar}} & \mathcal{W}_D \end{array}$$

¹For a related material on the deformation quantization of cotangent bundles with a different approach see [10]

Then the key point in this step is to find the subalgebra \mathcal{W}_D , which turns out to be the set of flat sections of a connection, and the correspondent map ϕ .

Now we proceed with the exposition in detail. We shall use the language of supermanifolds which makes it more simple and makes the nature of the objects used more transparent.

Consider a $(3n|n)$ -dimensional supermanifold $\mathcal{M} := \Omega^1 M \times_M TM \times_M \Pi(TM)$. (Where we have used the parity change operator Π .) For a coordinate system in $\Omega^1 M$ of the form $(x^1 \dots x^n, p_1 \dots p_n)$ we have an associated coordinate system on \mathcal{M} of the form

$$(x^1 \dots x^n, p_1 \dots p_n, y^1 \dots y^n, \psi^1 \dots \psi^n).$$

Definition 2.1 *The Weyl algebra \mathcal{W} on the supermanifold \mathcal{M} is the usual supercommutative algebra $\mathcal{O}_{\mathcal{M}}[[\hbar]]$, and a typical element of \mathcal{W} has locally the form*

$$a(x, p, y, \psi) = \sum_{k,p,r=0}^{\infty} \hbar^k a_{k,i_1 \dots i_p, j_1 \dots j_q}^{k_1 \dots k_r}(x) y^{i_1} \dots y^{i_p} p_{k_1} \dots p_{k_r} \psi^{j_1} \dots \psi^{j_q}, \quad (1)$$

where the tensor $a_{k,i_1 \dots i_p, j_1 \dots j_q}^{k_1 \dots k_r}$ is symmetric in the $i_1 \dots i_p$ and $k_1 \dots k_r$ indices and antisymmetric in the $j_1 \dots j_q$ indices.

There is a natural \star -product defined on \mathcal{W} given as follows

$$f \star_{\hbar} g = \exp\left(-\frac{i\hbar}{2}\left(\frac{\partial^2}{\partial y^a \partial \tilde{p}_a} - \frac{\partial^2}{\partial \tilde{y}^a \partial p_a}\right)\right) f(x, p, y, \psi) g(x, \tilde{p}, \tilde{y}, \psi)|_{p=\tilde{p}, y=\tilde{y}}$$

where $f, g \in \mathcal{W}$. This product is manifestly covariant and is easy to check it is associative. Note also that $f \star_{\hbar} g = g \star_{-\hbar} f$.

Then some auxiliary vector fields on \mathcal{M} are defined

$$\begin{aligned} i) \quad \delta &:= \psi^a \frac{\partial}{\partial y^a} \\ ii) \quad \delta^{\star} &:= y^a \frac{\partial}{\partial \psi^a} \\ iii) \quad d &:= \psi^a \frac{\partial}{\partial x^a} \\ iv) \quad \delta^{-1} a &:= \frac{\delta^{\star}}{p+q} a \\ v) \quad \partial a &:= \psi^i \partial_i a. \end{aligned}$$

where $a \in \mathcal{W}$ is an homogeneous element of order p in the variable y^a and of order q in the anticommutative variable ψ^a .

Lemma 2.2 *Let $a \in \mathcal{W}$ be an homogeneous element as in the last paragraph and define $a_{00} := a(x, p, 0, 0)$, then*

$$\begin{aligned} i) \quad \delta^2 a &= 0, \\ ii) \quad \delta^{\star 2} a &= 0, \\ iii) \quad \delta a &= -\frac{i}{\hbar} [p_j \psi^j, a]_{\star}, \\ iv) \quad a &= a_{00} + \frac{1}{p+q} (\delta \delta^{\star} a + \delta^{\star} \delta a) \end{aligned}$$

Proof is done by direct calculation, to illustrate we show the check for *iii*) which goes as follows

$$\begin{aligned}
-\frac{i}{\hbar}[p_i\psi^i, a]_* &= -\frac{i}{\hbar}(p_i\psi^i a + (\frac{-i\hbar}{2})(\frac{\partial^2}{\partial y^a \partial \tilde{p}_a} - \frac{\partial^2}{\partial \tilde{y}^a \partial p_a})p_i\psi^i a(x, \tilde{y}, \tilde{p}, \psi) \\
&\quad - (-1)^{\tilde{a}}(ap_i\psi^i + (\frac{-i\hbar}{2})(\frac{\partial^2}{\partial y^a \partial \tilde{p}_a} - \frac{\partial^2}{\partial \tilde{y}^a \partial p_a})a(x, y, p, \psi)\tilde{p}_i\psi^i)) \\
&= -\frac{i}{\hbar}(\frac{-i\hbar}{2})(-\psi^i \frac{\partial a}{\partial y^i} - (-1)^{\tilde{a}} \frac{\partial a}{\partial y^i} \psi^i) \\
&= \psi^i \frac{\partial a}{\partial y^i} \\
&= \delta a
\end{aligned}$$

□

Note that property *iv*) implies that for all $a \in \mathcal{W}$ there is a decomposition

$$a = \delta\delta^{-1}a + \delta^{-1}\delta a + a_{00}. \quad (2)$$

The local expression of ∂ is

$$\partial a = \psi^a(\frac{\partial}{\partial x^a} + \Gamma_{ab}^c p_c \frac{\partial}{\partial p_b} - \Gamma_{ab}^c y^b \frac{\partial}{\partial y_c})a. \quad (3)$$

Lemma 2.3 *It is possible to express ∂a as*

$$\partial a = da + \frac{i}{\hbar}[\Gamma, a]_*$$

for some $\Gamma \in \mathcal{W}$ of odd parity.

Proof. This can be shown as follows. Consider, for some constant α , the expression

$$da + [\alpha\Gamma, a]_* = da + i\hbar\alpha(-\frac{\partial\Gamma}{\partial y^b} \frac{\partial a}{\partial p_b} + \frac{\partial\Gamma}{\partial p_c} \frac{\partial a}{\partial y^c}) + O(\hbar^2).$$

Comparing this equation with equation (3) leads us to the equations

$$\psi^a \Gamma_{ab}^c p_c = -i\hbar \frac{\partial\alpha\Gamma}{\partial y^b}, \quad -\psi^a \Gamma_{ab}^c y^b = i\hbar \frac{\partial\alpha\Gamma}{\partial p_c}.$$

This implies that we must take $\Gamma = \Gamma_{ab}^c y^b p_c \psi^a$ and $\alpha = \frac{i}{\hbar}$ and the result follows. □

The auxiliary algebra is defined as the set $\mathcal{W}_D := \{a \in \mathcal{W} : Da = 0\}$, for some connection $D = \psi^a D_a$ that must satisfy the integrability condition $D^2 = 0$. It turns out that taking $D = \partial$ is not a good choice as the following proposition shows.

Proposition 2.4 *The integrability condition for the connection ∂ can be expressed as*

$$\partial^2 a = \frac{i}{\hbar}[R, a]_* \quad (4)$$

where $R := \frac{1}{2}\psi^b \psi^c R_{abc}^d p_d y^a$.

Proof.

$$\begin{aligned}
\frac{1}{2}[\partial, \partial]_* a &= \frac{1}{2}[\psi^a(\frac{\partial}{\partial x^a} + \Gamma_{ab}^c p_c \frac{\partial}{\partial p_b} - \Gamma_{ab}^c y^b \frac{\partial}{\partial y^c}), \psi^d(\frac{\partial}{\partial x^d} + \Gamma_{de}^f p_f \frac{\partial}{\partial p_e} - \Gamma_{de}^f y^e \frac{\partial}{\partial y^f})]_* a \\
&= \frac{1}{2}\psi^a \psi^d ((\frac{\partial \Gamma_{de}^f}{\partial x^a} - \frac{\partial \Gamma_{ae}^f}{\partial x^d}) p_f \frac{\partial}{\partial p_e} + (\frac{\partial \Gamma_{ae}^f}{\partial x^d} - \frac{\partial \Gamma_{de}^f}{\partial x^a}) y^e \frac{\partial}{\partial y^f} + \\
&\quad (\Gamma_{am}^f \Gamma_{de}^m - \Gamma_{ae}^m \Gamma_{dm}^f) p_f \frac{\partial}{\partial p_e} + (\Gamma_{ae}^m \Gamma_{dm}^f - \Gamma_{de}^m \Gamma_{am}^f) y^e \frac{\partial}{\partial y^f}) a
\end{aligned}$$

And since

$$R_{jkl}^i = \frac{\partial \Gamma_{jl}^i}{\partial x^k} - \frac{\partial \Gamma_{jk}^i}{\partial x^l} + \Gamma_{mk}^i \Gamma_{jl}^m - \Gamma_{ml}^i \Gamma_{jk}^m,$$

we have that

$$\partial^2 = \frac{1}{2} \psi^b \psi^c (R_{abc}^d p_d \frac{\partial}{\partial p_a} - R_{abc}^d y^a \frac{\partial}{\partial y^d})$$

and on the other hand we have

$$\frac{i}{\hbar} [R, a]_* = \frac{\partial R}{\partial y^a} \frac{\partial a}{\partial p_a} - \frac{\partial R}{\partial p_d} \frac{\partial a}{\partial y^d}.$$

This implies that we must take

$$R = \frac{1}{2} \psi^b \psi^c R_{abc}^d p_d y^a,$$

in order to have $\partial^2 = \frac{i}{\hbar} [R, \cdot]_*$. This completes the proof. \square

In other words the connection ∂ should be flat to fulfil the condition, which is far too restrictive. The way out is to define a new generalized connection of the form

$$Da = da + \frac{i}{\hbar} [-\psi^a p_a + \Gamma + \gamma, a]_* = \partial a + \frac{i}{\hbar} [-\psi^a p_a + \gamma, a]_*$$

where γ is

$$\gamma = \sum_{n=3}^{\infty} \Gamma_{i_1 \dots i_n, b}^a y^{i_1} \dots y^{i_n} p_a \psi^b, \quad (5)$$

to be determined to fulfill the integrability condition. Further calculation shows

$$\begin{aligned}
D^2 a &= \frac{i}{\hbar} [R, a]_* + \frac{i}{\hbar} \partial [-\psi^a p_a + \gamma, a]_* + \frac{i}{\hbar} [-\psi^a p_a + \gamma, \partial a]_* + (\frac{i}{\hbar})^2 [-\psi^a p_a + \gamma, [-\psi^a p_a + \gamma, a]_*]_* \\
&= \frac{i}{\hbar} [R, a]_* + \frac{i}{\hbar} [\partial \gamma, a]_* + (\frac{i}{\hbar})^2 \frac{1}{2} [[\psi^a p_a + \gamma, -\psi^a p_a + \gamma]_*, a]_* \\
&= \frac{i}{\hbar} [R + \partial \gamma - \delta \gamma + \frac{i}{\hbar} \gamma \star_{\hbar} \gamma, a]_*
\end{aligned}$$

Then the equivalent condition for having $D^2 = 0$ is

$$\delta \gamma = R + \partial \gamma + \frac{i}{\hbar} \gamma^2 \quad (6)$$

Proposition 2.5 γ is a solution of equation (6) if and only if

$$\gamma = \delta^{-1}R + \delta^{-1}(\partial\gamma + \frac{i}{\hbar}\gamma^2) \quad (7)$$

and the condition $\delta^{-1}\gamma = 0$ is fulfilled.

Fedosov's proof [6] can be applied here so we shall not include it.

Substitution of the general form of γ given in (5) into equation (7) leads to an iterative process with initial condition $\delta^{-1}R$. The first terms of the solution are

$$\gamma = \frac{1}{3}R_{abc}^d y^a y^b \psi^c p_d + \frac{1}{12}\partial_l R_{abc}^d y^l y^a y^b \psi^c p_d + \dots$$

The subalgebra \mathcal{W}_D is defined by the condition $Da = 0$, i.e.,

$$\delta a = \partial a + \frac{i}{\hbar}[\gamma, a]_* \quad (8)$$

Proposition 2.6 There is a one to one correspondence $\phi : \mathcal{O}_{\Omega^1 M}[[\hbar]] \rightarrow \mathcal{W}_D$.

Proof. This can be shown as follows. Condition (8) is equivalent to

$$a = a_{00} + \delta^{-1}(\partial a + [\frac{i}{\hbar}\gamma, a]_*). \quad (9)$$

The equivalence of these two equations is proved as in the Fedosov construction. Indeed, since $D^2a = 0$

$$\delta Da = \partial Da + [\frac{i}{\hbar}\gamma, Da]_* \quad (10)$$

On the other hand using (9) $\delta^{-1}Da = 0$ and so

$$Da = \delta^{-1}(\partial Da + [\frac{i}{\hbar}\gamma, Da]_*).$$

Solution of this equation by an iterative process implies that $Da = 0$. The converse assertion is trivial.

These are the first few terms of the solution for equation (9)

$$a = a_{00} + \partial_i a_{00} y^i + \frac{1}{2}\partial_i \partial_j a_{00} y^i y^j + \frac{1}{6}\partial_i \partial_j \partial_k a_{00} y^i y^j y^k - \frac{1}{12}R_{abc}^d y^a y^b p_d \frac{\partial a_{00}}{\partial p_c} + \dots$$

giving the one to one map $a_{00} \mapsto \phi(a_{00}) := a$. □

The star-product for $f_{00}, g_{00} \in \mathcal{O}_{\Omega^1 M}[[\hbar]]$ is finally defined as

$$f_{00} \star'_\hbar g_{00} = \phi^{-1}(f \star_\hbar g),$$

where $\phi^{-1}(f \star_\hbar g) = f \star_\hbar g(x, p, y, 0)|_{y=0}$. The star product \star'_\hbar inherits from \star_\hbar the natural shift from left to right multiplication $f_{00} \star_\hbar g_{00} = g_{00} \star_{-\hbar} f_{00}$, implying that our construction will not depend on our choice to use left ideals instead of right ideals.

We find our key result for this step

Theorem 2.7 *Let $f_{00}, g_{00} \in \pi^*(\mathcal{O}_M[[\hbar]])$, then*

$$f_{00} \star'_\hbar g_{00} = f_{00} g_{00}$$

Proof. Suppose that in equation (9) the starting condition $a_{00}(x, p)$ does not depend on p , therefore the commutator $[\gamma, a_{00}]$ vanishes and one can check that this happens for every step in the iterative solution. We are left then with the equation

$$a = a_{00}(x, p) + \delta^{-1}(\psi^i \partial_i a).$$

Solutions of this equation are

$$a = a_{00} + \sum_{i=1}^n \frac{1}{n!} \partial_{i_1} \dots \partial_{i_n} a_{00} y^{i_1} \dots y^{i_n}.$$

Star products of functions of this type are clearly just the usual commutative product. Now let f_{00}, g_{00} be two functions not depending on p then

$$\begin{aligned} f_{00}(x) \star'_\hbar g_{00}(x) &= \phi^{-1}(f \star_\hbar g) = \phi^{-1}(fg) \\ &= f_{00}(x) g_{00}(x). \end{aligned}$$

□

From this point we denote the star product on $\Omega^1 M$ simply as \star_\hbar .

Step 3 *Define the left ideal \mathcal{J}_l and compute the its normalizer \mathcal{N}_l .*

With the natural projection $\pi : \Omega^1 M \rightarrow M$ the ideal $I \subset \mathcal{O}_M[[\hbar]]$ can be lifted to $\Omega^1 M$ giving a set $\pi^*(I) \subset \mathcal{O}_{\Omega^1 M}$ which defines a left ideal

$$\mathcal{J}_l = \{\mathcal{O}_{\Omega^1 M}[[\hbar]] \star_\hbar \pi^*(I)\}.$$

Consider now the normalizer $\mathcal{N}_l \subset A_{\mathcal{M}}$ for the left ideal \mathcal{J}_l ,

$$\mathcal{N}_l = \{h \in \mathcal{O}_{\Omega^1 M}[[\hbar]] : \pi^*(I) \star_\hbar h \subset \mathcal{J}_l\}.$$

Clearly $\mathcal{J}_l \subset \mathcal{N}_l$ and moreover \mathcal{J}_l is a double sided ideal of \mathcal{N}_l , this is

$$f \star_\hbar s \in \mathcal{J}_l, \quad s \star_\hbar f \in \mathcal{J}_l,$$

for all $f \in \mathcal{N}_l, s \in \mathcal{J}_l$.

Step 4 *Take the quotient $\mathcal{Q}_X := \mathcal{N}_l / \mathcal{J}_l$. The result is a well defined non-commutative associative algebra which we call the quantum algebra of observables of X .*

Computing the normaliser of a one-sided ideal and taking the quotient to find a non-commutative ring is a rather common procedure which in general leads to conditions difficult to solve, but in our case, remarkably we have

Theorem 2.8 *The algebra \mathcal{Q}_X corresponds to the quotient solution space of a finite number of partial differential equations and it does not depend on the choice of generators of the ideal I .*

Proof. The key point of the proof is given by theorem (2.7). This implies that for any $g \in \mathcal{O}_{\Omega^1 M}$ the condition to be in the normalizer

$$\pi^{-1}(I) \star_{\hbar} g = 0 \text{ mod } \mathcal{J}_l$$

is equivalent to

$$\left(\sum_{i=1}^n \alpha_i \phi_i \right) \star_{\hbar} g = \sum_{i=1}^n \alpha_i \star_{\hbar} (\phi_i \star_{\hbar} g) = 0 \text{ mod } \mathcal{J}_l. \quad (11)$$

In other words to the condition that $\phi_i \star_{\hbar} g = 0 \text{ mod } \mathcal{J}_l$ for the n generators ϕ_i of $\pi^{-1}(I)$. Each equation being in fact a partial differential equation for functions in $\mathcal{O}_{\Omega^1 M}[[\hbar]]$. The independence of the choice of generators of the ideal follows trivially from the fact that any new set of generators $\tilde{\phi}_i$ can be rewritten as a combination of the original ones leading again to equation (11). \square

3 Examples

We shall develop next several examples of the explained technique when the ambient configuration space is \mathbb{R}^2 . The natural choice of star product is the Moyal product on $\Omega^1 \mathbb{R}^2$, given by

$$f \star_{\lambda} g = e^{\lambda \sum_{i=1}^2 \left(\frac{\partial}{\partial x_i} \frac{\partial}{\partial p_i} - \frac{\partial}{\partial \tilde{x}_i} \frac{\partial}{\partial \tilde{p}_i} \right)} f(x_1, x_2, p_1, p_2) g(\tilde{x}_1, \tilde{x}_2, \tilde{p}_1, \tilde{p}_2) |_{\tilde{x}=x, \tilde{p}=p}$$

with $f, g \in \mathcal{O}_{\Omega^1 \mathbb{R}^2}[[\lambda]]$. Clearly the moyal product of functions not depending on the momentum variables coincides with the pointwise product. We shall be using the following

Lemma 3.1 *Every analytic function $f(x_1, x_2, p_1, p_2) \in \Omega^1 \mathbb{R}^2$ can be uniquely decomposed as*

$$f(x_1, x_2, p_1, p_2) = \sum_{i,j=0}^{\infty} f_{ij}(p_1, p_2) \star_{\lambda} x_1^i \star_{\lambda} x_2^j \quad (12)$$

Proof. It is sufficient to note that

$$\begin{aligned} h(p_1, p_2) x_1^M x_2^N &= h(p_1, p_2) \star_{\lambda} x_1^M x_2^N \\ &- \sum_{n=1}^M \sum_{k=0}^{\min(N,n)} \frac{(-\lambda)^n}{(n-k)!k!} \frac{M!N!}{(M-n+k)!(N-k)!} x_1^{M-n+k} x_2^{N-k} \frac{\partial^n h}{\partial p_1^{(n-k)} \partial p_2^k} \end{aligned}$$

implying that the Taylor expansion can be re-expressed in terms of the Moyal product. \square

3.1 The cross

Consider now the analytic variety of the cross defined by the equation $x_1 x_2 = 0$. Any function on the quantum algebra $\mathcal{Q} = \mathcal{N}_l / \mathcal{J}_l$ can be expressed as

$$h(x_1, x_2, p_1, p_2) = h_0(p_1, p_2) + h_1(x_1, p_1, p_2) \star_{\lambda} x_1 + h_2(x_2, p_1, p_2) \star_{\lambda} x_2.$$

The left ideal is

$$\mathcal{J}_l = \{f *_{\lambda} x_1 x_2 : f \in \mathcal{O}_{\Omega^1 \mathbb{R}^2}\}.$$

The condition for the function h to be in the normalizer \mathcal{N}_l is $x_1 x_2 *_{\lambda} h \in \mathcal{J}_l$, i.e.,

$$\begin{aligned} 0 \bmod \mathcal{J}_l &= x_1 x_2 *_{\lambda} h_0 + x_1 x_2 *_{\lambda} h_1 *_{\lambda} x_1 + x_1 x_2 *_{\lambda} h_2 *_{\lambda} x_2 \\ &= [x_1 x_2, h_0] + [x_1 x_2, h_1] *_{\lambda} x_1 + [x_1 x_2, h_2] *_{\lambda} x_2 \\ &= 2\lambda \frac{\partial^2 h_0}{\partial p_1 \partial p_2} + \left(\frac{\partial h_0}{\partial p_2} + \lambda \frac{\partial^2 h_1}{\partial p_1 \partial p_2} + x_1 \frac{\partial h_1}{\partial p_2} \right) *_{\lambda} x_1 \\ &\quad + \left(\frac{\partial h_0}{\partial p_1} + \lambda \frac{\partial^2 h_2}{\partial p_1 \partial p_2} + x_2 \frac{\partial h_2}{\partial p_1} \right) *_{\lambda} x_2 \end{aligned}$$

Solutions have the form

$$\begin{aligned} h_1(x_1, p_1, p_2) &= -\frac{h_0^2(p_2)}{x_1} + \xi(x_1, p_2) e^{-\frac{x_1 p_1}{\lambda}} + a(x_1, p_1) \\ h_2(x_2, p_1, p_2) &= -\frac{h_0^1(p_1)}{x_2} + \zeta(x_2, p_1) e^{-\frac{x_2 p_2}{\lambda}} + b(x_2, p_2). \end{aligned}$$

where $h_0^1(p_1), h_0^2(p_2)$ are arbitrary functions. The solutions of the form $\frac{h_0^2(p_2)}{x_1}, \frac{h_0^1(p_1)}{x_2}$ are not defined on the cross, therefore we do not consider them, similarly the terms $\xi(x_1, p_2) e^{-\frac{x_1 p_1}{\lambda}}$ and $\zeta(x_2, p_1) e^{-\frac{x_2 p_2}{\lambda}}$ must be rejected as they are not meromorphic in λ . (However such solutions may have a physical interpretation which we hope to elucidate later.)

We have then a family of functions for the quantum algebra of the cross given by

$$\mathcal{Q}_C := \{h \in \mathcal{O}_{\Omega^1 \mathbb{R}^2}[[\lambda]] : h(x_1, x_2, p_1, p_2) = a(x_1, p_1) *_{\lambda} x_1 + b(x_2, p_2) *_{\lambda} x_2\}$$

Computing the Moyal product of two elements $h, \tilde{h} \in \mathcal{Q}_C$ and eliminating terms with the factor $x_1 x_2$ we find

$$\begin{aligned} h *_{\lambda} \tilde{h} &= (a *_{\lambda} x_1 + b *_{\lambda} x_2) *_{\lambda} (\tilde{a} *_{\lambda} x_1 + \tilde{b} *_{\lambda} x_2) \\ &= a *_{\lambda} x_1 *_{\lambda} \tilde{a} + b *_{\lambda} x_2 *_{\lambda} \tilde{b}. \end{aligned}$$

In other words the quantum algebra of the cross has then elements of the form

$$h = (a(x_1, p_1), b(x_2, p_2))$$

where a, b are arbitrary functions and the noncommutative product is

$$h \star_C \tilde{h} = (a *_{\lambda} x_1 *_{\lambda} \tilde{a}, b *_{\lambda} x_2 *_{\lambda} \tilde{b}).$$

3.2 The double line

Consider now the analytic space of the double line defined by the equation $x_2^2 = 0$. Functions of the quantum algebra $\mathcal{N}_l/\mathcal{J}_l$ can be represented as

$$h(x_1, x_2, p_1, p_2) = h_0(x_1, p_1, p_2) + h_1(x_1, p_1, p_2) *_{\lambda} x_2.$$

The condition for h to be in the normalizer of the left ideal $\mathcal{J}_l = \{f *_{\lambda} x_2^2; f \in \mathcal{O}_{\Omega^1 \mathbb{R}^2}\}$ is

$$0 \bmod \mathcal{J}_l = [x_2^2, h_0] + [x_2^2, h_1] *_{\lambda} x_2$$

Leading to the differential equation

$$x_2 \frac{\partial h_0}{\partial p_2} + \lambda x_2 \frac{\partial^2 h}{\partial p_2^2} + \lambda^2 \frac{\partial^3 h}{\partial p_2^3} = 0$$

whose general solution is

$$h(x_1, x_2, p_1, p_2) = a(x_1, p_1) + b(x_1, p_1)p_2 + \left(c(x_1, p_1) + d(x_1, p_1)p_2 - \frac{b(x_1, p_1)}{2\lambda} p_2^2 \right) *_{\lambda} x_2.$$

The product of two of these functions can be represented as a matrix star-product denoted as \star , in the following way

$$\phi(h) := \begin{pmatrix} a + 2\lambda d & b \\ 2\lambda c & a \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Then

$$\begin{aligned} \phi(h) \star \phi(\tilde{h}) &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} *_{\lambda} \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \\ &= \begin{pmatrix} A *_{\lambda} \tilde{A} + B *_{\lambda} \tilde{C} & A *_{\lambda} \tilde{B} + B *_{\lambda} \tilde{D} \\ C *_{\lambda} \tilde{A} + D *_{\lambda} \tilde{C} & C *_{\lambda} \tilde{B} + D *_{\lambda} \tilde{D} \end{pmatrix} \end{aligned}$$

is equal to $\phi(h *_{\lambda} \tilde{h})$.

3.3 Line with a double point

We shall proceed now with the quantization of the line with a double point defined as the quotient of the quantum algebra of the double line quotient by the ideal generated by $x_1 x_2 = 0$ which has the form

$$\begin{aligned} \phi(h *_{\lambda} x_1 x_2) &= \phi(h) \star \phi(x_1 x_2) \\ &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} *_{\lambda} \begin{pmatrix} 0 & 0 \\ x_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} B *_{\lambda} x_1 & 0 \\ D *_{\lambda} x_1 & 0 \end{pmatrix}, \end{aligned}$$

where B, D are arbitrary functions depending on x_1, p_1 . The corresponding normalizer will be the set of solutions of the equation

$$\begin{pmatrix} 0 & 0 \\ x_1 & 0 \end{pmatrix} *_{\lambda} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 \bmod \mathcal{J}_l, \quad (13)$$

where a, b, c, d are a different set of arbitrary functions depending on x_1, p_1 . This implies in turn that $b = 0$ since the only solutions of the differential equation $x_1 b + \lambda \partial_{p_1} b = 0$ are

not in the space of acceptable formal functions. Then $a(x_1, p_1)$ must be a solution of the equation

$$x_1 *_{\lambda} a = 0 \pmod{D *_{\lambda} x_1}.$$

Decomposing $a = \sum_{i=0}^{\infty} a_i(p_1) *_{\lambda} x_1^i$ and factoring out we can rewrite this as

$$x_1 *_{\lambda} a_0(p_1) = 0 \pmod{D *_{\lambda} x_1}.$$

Lemma 3.2

$$A(x, p) *_{\lambda} B(p) = \sum_{n=0}^{\infty} (2\lambda)^n A_n(p) B^{(n)}(p) \pmod{D *_{\lambda} x}$$

where $A(x, p) = \sum A_n(p) *_{\lambda} x^n$ and $B^{(n)}$ is the n th derivative of B .

Proof.

$$\begin{aligned} \sum_{i=0}^{\infty} A_i(p) *_{\lambda} x^{i-1} *_{\lambda} x *_{\lambda} B(p) &= \sum_{i=0}^{\infty} A_i(p) *_{\lambda} x^{i-1} *_{\lambda} 2\lambda B^{(1)} \pmod{D *_{\lambda} x} \\ &= \sum_{i=0}^{\infty} (2\lambda)^i A_i(p) *_{\lambda} B^{(i)}(p) \end{aligned}$$

□

In particular this implies that the second equation means that $a = k$ for some constant. Then the normalizer has the form

$$\begin{pmatrix} k & 0 \\ c(x_1, p_1) & d(x_1, p_1) \end{pmatrix}$$

The last step is to factor out the members of the left ideal from this normalizer. This means to take

$$c(x_1, p_1) \pmod{D *_{\lambda} x_1}.$$

The resulting quantum algebra for the line with a double point is the set of matrices of the form

$$\begin{pmatrix} k & 0 \\ c(p_1) & d(x_1, p_1) \end{pmatrix}$$

with multiplication law

$$\begin{aligned} &\begin{pmatrix} k & 0 \\ c(p_1) & d(x_1, p_1) \end{pmatrix} \star' \begin{pmatrix} \tilde{k} & 0 \\ \tilde{c}(p_1) & \tilde{d}(x_1, p_1) \end{pmatrix} \\ &= \begin{pmatrix} k\tilde{k} & 0 \\ \tilde{k}c(p_1) + \sum_{n=0}^{\infty} (2\lambda)^n d_n(p_1) \tilde{c}^{(n)}(p_1) & d(x_1, p_1) *_{\lambda} \tilde{d}(x_1, p_1) \end{pmatrix} \end{aligned}$$

where we have used lemma 3.2.

3.4 The doubly fattened circle

Let us consider now the space associated to the ideal generated by $x_1^2 + x_2^2$, that we refer to as the "doubly fattened" circle. Although the zero set of this polynomial is just the origin the resulting quantum algebra, as we shall show, is nontrivial. Any function in this quantum algebra can be represented as

$$h(x_1, x_2, p_1, p_2) = h_0(x_1, p_1, p_2) + h_1(x_1, p_1, p_2) *_{\lambda} x_2$$

The left ideal is

$$\mathcal{J}_l = \{f *_{\lambda} (x_1^2 + x_2^2) : f \in \mathcal{O}_{\Omega^1 \mathbb{R}^2}[[\lambda]]\}.$$

The condition for a function $h(x_1, x_2, p_1, p_2)$ to be in the normalizer is

$$\begin{aligned} 0 \bmod \mathcal{J}_l &= (x_1^2 + x_2^2) *_{\lambda} h \\ &= -\frac{\partial h_1}{\partial p_2} x_1^2 + 2\lambda x_1 \frac{\partial^2 h_1}{\partial p_1 \partial p_2} - \lambda^2 \frac{\partial^3 h_1}{\partial p_1^2 \partial p_2} + \lambda \frac{\partial^2 h_0}{\partial p_2^2} + x_1 \frac{\partial h_0}{\partial p_1} \\ &\quad + \left(\frac{\partial h_0}{\partial p_2} + \lambda \frac{\partial^2 h_1}{\partial p_2^2} + x_1 \frac{\partial h_1}{\partial p_1} \right) *_{\lambda} x_2 \end{aligned}$$

Manipulations in the last equation lead to the condition

$$x_1^2(\Delta h_1) - \lambda^2 \frac{\partial^2}{\partial p_2^2}(\Delta h_1) = 0,$$

where $\Delta = \partial_{p_1}^2 + \partial_{p_2}^2$, is the Laplacian operator. Therefore the two dimensional spherical harmonics give a family of solutions h_1 .

3.5 Conclusion and Acknowledgments

We have shown a way to define non-commutative associative products to analytic spaces immersed in analytic manifolds. The procedure works for smooth spaces and more remarkably it works for singular spaces. This opens new possibilities in the field of deformation quantization and leaves many open questions. One such question is how the singularity affects the resulting algebra, or more generally how the singularity type affects it. Questions like these can be studied through the introduction of parameters deforming the analytic spaces which will enter the partial differential equations defining the associated algebras. These equations turn out to be complex in most cases. New techniques need to be developed to find the representation of the algebras to make possible the study of such questions.

Another interesting possibility is to develop a comparative study of the different quantization programmes. The common goal of all such programmes is to give a quantized version of classical physical systems. Naturally for any given classical system there should be a unique physical quantum version, thus one would expect the different approaches to be equivalent in some sense. Having an affirmative answer to this would give a hint of some deep mathematical relations between the different approaches.

Another important line of work is to develop proper physical applications of the programme. A rather interesting question to be studied is to find out how the singularity affects the physics of the space.

We expect to elucidate these and other questions in the future.

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